

Coarse-grained Monte Carlo simulations of the phase transition of Potts model on weighted networks

Chuansheng Shen^{1,2}, Hanshuang Chen¹, Zhonghuai Hou^{1,*} and Houwen Xin¹

¹ Hefei National Laboratory for Physical Sciences at Microscales & Department of Chemical Physics,

University of Science and Technology of China, Hefei, 230026, China

²Department of Physics, Anqing Teachers College, Anqing, 246011, China

Developing effective coarse grained (CG) approach is a promising way for studying dynamics on large size networks. In the present work, we have proposed a strength-based CG (*s*-CG) method to study critical phenomena of the Potts model on weighted complex networks. By merging nodes with close strength together, the original network is reduced to a CG-network with much smaller size, on which the CG-Hamiltonian can be well-defined. In particular, we make error analysis and show that our strength-based CG approach satisfies the condition of statistical consistency, which demands that the equilibrium probability distribution of the CG-model matches that of the microscopic counterpart. Extensive numerical simulations are performed on scale-free networks, without or with strength-correlation, showing that this *s*-CG approach works very well in reproducing the phase diagrams, fluctuations, and finite size effects of the microscopic model, while the *d*-CG approach proposed in our recent work [Phys. Rev. E 82, 011107(2010)] does not.

PACS numbers: 05.50.+q, 89.75.Hc, 05.10.-a

I. INTRODUCTION

In the last two decades, we have witnessed dramatic advances in complex networks research, which has been one of the most active topics in statistical physics and closely related disciplines [1–5]. The central issue in this field is to study how the topology of networks influences dynamics, such as phase transition, self-organized criticality and epidemic spreading, etc. Usually, Monte Carlo (MC) simulations [6] have been widely used to study such dynamics. However, the sizes of many real-world networks are very large, such as human brain composed of about 10^{11} neurons and 10^{14} synapses [7], and thereby brute-force simulations are quite expensive and sometimes even become impossible. Phenomenological models, such as mean-field description, may capture certain properties of the system, but often ignore microscopic details and fluctuation effects which may be important near some critical points. Therefore, a promising way to bridge the gap between the microscopic details and system level behaviors is to develop coarse-grained (CG) approaches, aiming at significantly reducing the degree of freedom while properly preserving the microscopic information of interest.

Recently, several CG approaches have been proposed in the literature. Renormalization transformation has been used to reduce the size of self-similar networks while preserving the most relevant topological properties of the original ones [8–11]. Gfeller and Rios proposed spectral decomposition technique to obtain a CG-network which can reproduce the random walk and synchronization dynamics of the original network [12]. Kevrekidis et al. developed equation-free multiscale computational meth-

ods to accelerate simulation using a coarse time-stepper [13], which has been successfully applied to study the CG dynamics of oscillator networks [14], gene regulatory networks [15], and adaptive epidemic networks [16]. Nevertheless, none of the works mentioned above has considered critical phenomena in complex networks, which has been a frontier topic in the context of network science [17].

Very recently, we have proposed a degree-based CG (*d*-CG) approach to study the critical phenomena of the Ising model and the SIS-epidemic model in unweighted networks [18]. A local mean field (LMF) scheme was introduced to generate the CG network from the microscopic one. Specifically, we have proposed a so-called condition of statistical consistency (CSC) that the CG-model should satisfy to guarantee the validity of the CG-approach. We showed that the CSC can be exactly fulfilled if we merge nodes with the same degree together. Extensive numerical simulations showed that our *d*-CG approach does work very well to reproduce the phase transition behaviors of the original network, including the critical point and the fluctuation properties, but with much less computational efforts. Our method also makes it feasible to investigate the finite size effects of both models, which should be much more expensive and even forbidden if we use brute-force methods. However, this *d*-CG approach can only apply to binary networks, i.e., each of the link in the network either exists or not, but with no weight. As we know, many real-world networks are intrinsically weighted, with their links having diverse strengths. Examples include the collaboration networks [19–21], airport networks [22, 23], metabolic networks [24] predator-prey relationship networks [25], and so on. Therefore, a straightforward question is: Can we use CG approaches to study the critical phenomena in weighted networks?

To answer this question, in the present work, we have

*Electronic address: hzhlj@ustc.edu.cn

considered the critical phenomena of the Potts model in weighted complex networks. The Potts model is related to a number of important topics in statistical and mathematical physics [26, 27] and was successfully applied to neural networks, multiclass classification problems, graph coloring problem, and so on. It contains a system of coupled nodes, each of which has p possible states. Only when two nodes are in the same state, they have pairwise interactions. With the increment of temperature, the Potts model undergoes an order-disorder phase transition at some critical temperature. For $p = 2$, Potts model is equivalent to the well-known Ising model. Instead of the d -CG scheme, we have proposed a strength-based CG (s -CG) approach, where those nodes with *similar* strength are merged together to form a CG-node. Note that in weighted networks, it is unpractical to merge nodes with exactly the same strength together. By detailed analysis of the discrepancy between the Hamiltonian of a CG configuration and that of its corresponding microscopic configurations, we show that the s -CG approach can approximately satisfy the CSC defined on weighted networks. Extensive numerical simulations are performed on scale-free(SF) networks, without or with strength-correlation, showing that our s -CG approach works very well in reproducing the phase diagrams, fluctuations, and finite size effects of the microscopic model, while the simple d -CG does not. Compared to our previous work [18], the present study step forward several important steps. First of all, we should note that s -CG is a brand new method compared to d -CG and the latter cannot apply to weighted networks, although they share some similar ideas. Secondly, weighted networks are of more ubiquitous importance than binary unweighted ones, thus the s -CG approach should find more applications. What is more, we have extended the study from the simple two-state Ising model to a more general one, the multi-state Potts model. In addition, we have performed error analysis in the present study, which clearly demonstrates the robustness of our approaches.

II. COARSE GRAINING PROCEDURE

A. CG Potts Model

In this paper, we consider the p -states Potts model on a weighted network consisted of N nodes, whose Hamiltonian is given by

$$H = - \sum_{i < j} w_{ij} \delta_{\alpha_i, \alpha_j}, \quad (1)$$

where w_{ij} is the weight on the edge connecting a pair of nodes i and j ($w_{ij} = 0$ if the nodes i and j are not connected). $\alpha_i (= 1, \dots, p)$ denotes the state of node i , $\delta_{\alpha_i, \alpha_j} = 1$ if $\alpha_i = \alpha_j$ and 0 otherwise.

To setup the CG-Potts model, one needs to obtain the CG-Hamiltonian defined on the CG-network, fol-

lowed by CG-MC simulations to study the dynamic behaviors. The CG-network is simply obtained by node-merging, i.e., q_μ nodes within the original micro-network are merged into a single CG-node C_μ , where $\mu = 1, \dots, N^c$ labels the CG-node and N^c is size of the CG-network. Following the LMF scheme used in Ref. [18], the weight of link between two CG nodes μ and ν reads,

$$\bar{w}_{\mu\nu} = \begin{cases} \frac{2}{q_\mu(q_\mu-1)} \sum_{i,j \in C_\mu; i < j} w_{ij} & \text{for } \mu = \nu, \\ \frac{1}{q_\mu q_\nu} \sum_{i \in C_\mu, j \in C_\nu} w_{ij} & \text{for } \mu \neq \nu. \end{cases} \quad (2)$$

The CG-Hamiltonian \bar{H} can be readily obtained,

$$\bar{H} = \bar{H}_1 + \bar{H}_2$$

where

$$\bar{H}_1 = - \sum_\mu \bar{w}_{\mu\mu} \sum_\alpha \frac{\eta_{\mu,\alpha}(\eta_{\mu,\alpha} - 1)}{2} \quad (3a)$$

$$\bar{H}_2 = - \sum_{\mu, \nu (> \mu)} \bar{w}_{\mu\nu} \sum_\alpha \eta_{\mu,\alpha} \eta_{\nu,\alpha} \quad (3b)$$

Herein, $\bar{H}_1(\bar{H}_2)$ denote CG interactions inside(among) the CG-nodes, respectively. $\eta_{\mu,\alpha}$ stands for the number of α -state micro-nodes inside C_μ . Since there are $\frac{\eta_{\mu,\alpha}(\eta_{\mu,\alpha}-1)}{2}$ possible distinct pairs of α -state micro-nodes inside C_μ , and each pair has a weighted coupling $w_{\mu\mu}$, the CG-interactions among all the α -state nodes inside C_μ is given by

$$\bar{H}_{\mu,1}^{(\alpha)} = -\bar{w}_{\mu\mu} \frac{\eta_{\mu,\alpha}(\eta_{\mu,\alpha} - 1)}{2}.$$

Summation this over all CG-nodes μ and states α gives the result in Eq.(3a). Eq.(3b) can be interpreted in a similar way. Note that Eq.(3) are closed at the CG level, i.e., as long as one has constructed the CG-network, \bar{w} and \bar{H} are then both well defined, based on which one can perform CG-MC simulations without going back to the micro-level.

B. CSC: Condition of statistical consistency

The above procedure tells us how to calculate the CG-Hamiltonian if we already have the CG-network. However, which q_μ nodes are merged together to form a CG-node C_μ is yet not determined. Generally speaking, one may construct the CG-network deliberately, for instance, one may simply generate N^c values, q_μ obeying $\sum_{\mu=1}^{N^c} q_\mu = N$ and then just randomly merge q_μ micro-nodes to form C_μ . Therefore, an important question arises: How to guarantee that the CG-model can reproduce the dynamics of the corresponding microscopic model correctly?

We address this problem by extending the so-called CSC as proposed in [18]. We introduce $\vec{\eta}_\mu =$

$\{\eta_{\mu,\alpha}\}_{\alpha=1,\dots,p}$ to denote the state of C_μ and $\vec{\eta} = \{\vec{\eta}_\mu\}_{\mu=1,\dots,N^c}$ to denote the configuration of the CG-network. Note that a given CG configuration $\vec{\eta}$ corresponds to many microscopic configurations, which defines the degeneracy factor $g(\vec{\eta})$. In the equilibrium state of the CG-model, the probability of finding a given CG-configuration $\vec{\eta}$ is given by the canonical distribution, i.e.,

$$p_{\text{CG}}(\vec{\eta}) = g(\vec{\eta}) e^{-\bar{H}/k_B T} / \bar{Z},$$

where $\bar{Z} = \sum_{\vec{\eta}} p_{\text{CG}}(\vec{\eta})$ is the CG partition function. It is important to note, however, that $p(\vec{\eta})$ can be calculated exactly from the equilibrium distribution of the micro-model,

$$p_{\text{micro}}(\vec{\eta}) = \sum' e^{-H/k_B T} / Z,$$

where Z is the partition function of the micro-model, and the prime means summation over all the microscopic configurations that contribute to $\vec{\eta}$. Since we are interested in the equilibrium phase transition behavior of the Potts model, we thus assert that for the CG-model to be statistically consistent with the micro-model, $p_{\text{CG}}(\vec{\eta})$ and $p_{\text{micro}}(\vec{\eta})$ must be equal, i.e., the CSC reads

$$g(\vec{\eta}) e^{-\bar{H}/k_B T} / \bar{Z} = \sum' e^{-H/k_B T} / Z. \quad (4)$$

C. s -CG Scheme and error analysis

In the present work, we propose a s -CG scheme to construct the CG-network, i.e., nodes with same or similar strengths are merged together to form a CG-node, where the strength s_i of node i is defined as $s_i = \sum_j w_{ij}$ [22, 28]. In the following, we will show that if nodes inside each CG-node have same strengths, the CSC will hold exactly within the ANA. In addition, if the strengths within C_μ are nearly the same, the CSC can also hold approximately.

In the literature, ANA [17, 29–31] has been widely used to study the ensemble averaged dynamics of complex networks and proved to be successful. ANA assumes that one can replace the dynamics on a given network by that on a weighted fully connected graph with connectivity $A_{ij} = d_i d_j / (DN)$, where d_i (d_j) denotes the degree of node i (j) and D is the mean degree of the network. Analogously, in weighted networks link weight can be expressed as

$$w_{ij} = s_i s_j / (SN) \quad (5)$$

where S is the mean strength of the network. Substituting Eq.(5) into Eq.(2), the adjacency matrix of the CG-network now reads,

$$\begin{aligned} \bar{w}_{\mu\mu} &= \frac{2}{q_\mu(q_\mu - 1)} \sum_{i < j \in C_\mu} \frac{(S_\mu + \delta s_i)(S_\mu + \delta s_j)}{SN} \\ &= \frac{S_\mu^2}{SN} (1 - \Omega_\mu) \end{aligned} \quad (6a)$$

$$\begin{aligned} \bar{w}_{\mu\nu} &= \frac{1}{q_\mu q_\nu} \sum_{i \in C_\mu, j \in C_\nu} \frac{(S_\mu + \delta s_i)(S_\nu + \delta s_j)}{SN} \\ &= \frac{1}{SN} \sum_{i \in C_\mu, j \in C_\nu} S_\mu S_\nu = \frac{S_\mu S_\nu}{SN} \end{aligned} \quad (6b)$$

Herein, we have written $s_i = S_\mu + \delta s_i$, with $S_\mu = \frac{1}{q_\mu} \sum_{i \in C_\mu} s_i$ being the mean strength within C_μ . $\Omega_\mu = \frac{\langle \delta s^2 \rangle_\mu}{S_\mu^2 (q_\mu - 1)}$ where $\langle \delta s^2 \rangle_\mu = \frac{1}{q_\mu} \sum_{i \in C_\mu} (\delta s_i)^2$ is the variance of strength within C_μ . In the first equation, we have used the fact that $(\sum_{i \in C_\mu} \delta s_i)^2 = 2 \sum_{i < j \in C_\mu} \delta s_i \delta s_j + \sum_{i \in C_\mu} (\delta s_i)^2 = 0$. The second equation holds simply because $\sum_{i \in C_\mu, j \in C_\nu} \delta s_i \delta s_j = (\sum_{i \in C_\mu} \delta s_i)(\sum_{j \in C_\nu} \delta s_j) = 0$. Substituting Eq.(6) into Eq.(3), we can get

$$\bar{H}_1 = -\frac{1}{SN} \sum_\mu S_\mu^2 (1 - \Omega_\mu) \sum_\alpha \frac{\eta_{\mu,\alpha}(\eta_{\mu,\alpha} - 1)}{2} \quad (7a)$$

$$\bar{H}_2 = \frac{1}{SN} \sum_{\mu, \nu (> \mu)} S_\mu S_\nu \sum_\alpha \eta_{\mu,\alpha} \eta_{\nu,\alpha}, \quad (7b)$$

To compare the CG-Hamiltonian with the microscopic one, we now group the micro-nodes with same state α inside C_μ as $C_{\mu,\alpha}$. Clearly, the size of $C_{\mu,\alpha}$ is $\eta_{\mu,\alpha}$. As in Eq.(3), we can also split the micro-Hamiltonian H into two parts ,

$$H = H_1 + H_2 \quad (8)$$

where H_1 and H_2 denote energy contributions from intra and inter the CG-nodes respectively. With ANA, and noting the fact only nodes with same states have interactions at the micro-level, one has

$$H_1 = - \sum_\mu \sum_\alpha \sum_{i < j \in C_{\mu,\alpha}} \frac{s_i s_j \eta_{\mu,\alpha}(\eta_{\mu,\alpha} - 1)}{SN} \quad (9a)$$

$$H_2 = - \sum_{\mu, \nu (> \mu)} \sum_\alpha \sum_{i \in C_{\mu,\alpha}, j \in C_{\nu,\alpha}} \frac{s_i s_j}{SN} \eta_{\mu,\alpha} \eta_{\nu,\alpha} \quad (9b)$$

Following similar steps to obtain Eq.(7), we may also write $s_i = S_{\mu,\alpha} + \delta s_i$ (here node i belongs to the group

$C_{\mu,\alpha}$) and Eqs.(9) change to

$$\begin{aligned} H_1 &= - \sum_{\mu} \sum_{\alpha} \sum_{i < j \in C_{\mu,\alpha}} \frac{(S_{\mu,\alpha} + \delta s_i)(S_{\mu,\alpha} + \delta s_j)}{SN} \times \\ &\quad \frac{\eta_{\mu,\alpha}(\eta_{\mu,\alpha} - 1)}{2} \\ &= - \frac{1}{SN} \sum_{\mu} \sum_{\alpha} S_{\mu,\alpha}^2 (1 - \Omega_{\mu,\alpha}) \frac{\eta_{\mu,\alpha}(\eta_{\mu,\alpha} - 1)}{2} \end{aligned} \quad (10a)$$

$$\begin{aligned} H_2 &= - \sum_{\mu, \nu (> \mu)} \sum_{\alpha} \sum_{i \in C_{\mu,\alpha}, j \in C_{\nu,\alpha}} \frac{(S_{\mu,\alpha} + \delta s_i)(S_{\nu,\alpha} + \delta s_j)}{SN} \times \\ &\quad \eta_{\mu,\alpha} \eta_{\nu,\alpha} \\ &= - \frac{1}{SN} \sum_{\mu, \nu (> \mu)} \sum_{\alpha} \eta_{\mu,\alpha} \eta_{\nu,\alpha} S_{\mu,\alpha} S_{\nu,\alpha} \end{aligned} \quad (10b)$$

Here $\Omega_{\mu,\alpha} = \frac{\langle \delta s^2 \rangle_{\mu,\alpha}}{S_{\mu,\alpha}^2 (\eta_{\mu,\alpha} - 1)}$ where $\langle \delta s^2 \rangle_{\mu,\alpha} = \frac{1}{\eta_{\mu,\alpha}} \sum_{i \in C_{\mu,\alpha}} (\delta s_i)^2$ is the variance of strength within the group of nodes $C_{\mu,\alpha}$.

Comparing Eq.(7) with Eq.(10), the discrepancy between the CG-Hamiltonian and the micro-Hamiltonian, is given by

$$\bar{H}_1 - H_1 = - \frac{1}{SN} \sum_{\mu} \sum_{\alpha} \frac{\eta_{\mu,\alpha}(\eta_{\mu,\alpha} - 1)}{2} \times [S_{\mu}^2 (1 - \Omega_{\mu}) - S_{\mu,\alpha}^2 (1 - \Omega_{\mu,\alpha})] \quad (11a)$$

$$\bar{H}_2 - H_2 = - \sum_{\mu, \nu} \sum_{\alpha} \frac{\eta_{\mu,\alpha} \eta_{\nu,\alpha} (S_{\mu} S_{\nu} - S_{\mu,\alpha} S_{\nu,\alpha})}{SN} \quad (11b)$$

Obviously, for the exact s -CG algorithm where all the nodes inside a given CG-node have same strengths, $\Omega_{\mu} = \Omega_{\mu,\alpha} = 0 \forall (\mu, \alpha)$ and $S_{\mu} = S_{\mu,\alpha}$, hence $\bar{H}_1 = H_1$ and $\bar{H}_2 = H_2$. In this case, all those microscopic configurations contributing to a CG-configuration $\vec{\eta}$ have exactly the same Hamiltonian H , which also equals to the CG-Hamiltonian \bar{H} . Since the constrained summation \sum' contains exactly $g(\vec{\eta})$ items, the numerators on both sides of Eq.(4) are exactly equal, i.e., $g(\vec{\eta}) e^{-\bar{H}/k_B T} = \sum' e^{-H/k_B T}$. Since we can also write the microscopic partition function as $Z = \sum_{\vec{\eta}} (\sum' e^{-H/k_B T})$, it is readily to show that the two partition functions equal, $\bar{Z} = Z$. Therefore, the CSC, Eq.(4), exactly holds.

However, we should note that for a weighted network, the exact s -CG method is not practical, since the strength of a given node is generally not an integer. Therefore, usually one can only merge nodes with close strength together. Let us analyze Eq.(11) again. The factor Ω_{μ} scales as $\frac{\langle \delta s^2 \rangle_{\mu}}{S_{\mu}^2 q_{\mu}}$, hence if we merge many nodes with similar strengths together, $\Omega_{\mu} \ll 1$ is expected to be true. One may also expect that $\Omega_{\mu,\alpha} \ll 1$ for the same reason. Therefore, the discrepancy between \bar{H} and H mainly depends on the difference between S_{μ} and $S_{\mu,\alpha}$. Here, we note that the nodes with α -state flip with time during the simulation. In the equilibrium state, one expects that $C_{\mu,\alpha}$ may scan throughout C_{μ} for many times,

such that $S_{\mu,\alpha}$ averaged over time is close to S_{μ} . Hence $(\bar{H} - H)/H$ averaged over long time could be small. Note that if we merge nodes randomly, $\Omega_{\mu} \ll 1$ and $\Omega_{\mu,\alpha} \ll 1$ will be violated and the above reasoning should fail. We thus conclude that the practical s -CG approach, by merging nodes with similar strength together, can satisfy the CSC approximately.

III. NUMERICAL RESULTS

To show the validity of our s -CG approach, we perform extensive simulations on weighted SF networks. SF networks are much heterogeneous and serve as better candidates to test our method than other homogeneous networks, such as small-world or random networks (other types of complex networks have also been investigated, the qualitative results are the same and not shown here). We first generate a regular (unweighted) SF network by using the Barabási-Albert (BA) model [32] with power-law degree distribution $P(k) \sim k^{-3}$. To convert this unweighted SF network into a weighted one, we use the algorithm as proposed in Ref. [33]: The weight of a link between node i and j ($1 \leq i, j \leq N$) is given by $w_{ij} = (\frac{i}{N} + \frac{j}{N})^{\theta}/2$, where θ is a tunable parameter. Note that $\theta = 0$ corresponds to an unweighed network.

The MC simulation at the microscopic level follows standard Metropolis dynamics: At each step, a micro-node is randomly selected and its state is randomly updated with an acceptance probability $\min(1, e^{-\Delta H/k_B T})$, where ΔH is the associated change of the micro-Hamiltonian, k_B is the Boltzmann constant, T is the temperature. In the present work, we set $k_B = 1$. Similarly, during each CG-MC step, a CG-node C_{μ} is randomly chosen with probability proportional to its size q_{μ} . The probability for the process that an α -node changes to a β -node, with correspondingly $\eta_{\mu,\alpha} \rightarrow \eta_{\mu,\alpha} - 1$ and $\eta_{\mu,\beta} \rightarrow \eta_{\mu,\beta} + 1$, is given by $\eta_{\mu,\alpha} \min(1, e^{-\Delta \bar{H}/k_B T})$, where $\Delta \bar{H}$ is the change of CG-Hamiltonian during this process. Since N^c can be much smaller than N , the CG-MC is expected to be much faster and memory-saving than the micro-level MC simulation.

The collective state of the system is described by the total magnetic moment $M = \frac{1}{2N} \sum_{\mu,\alpha} |M_{\mu,\alpha}|$, where

$M_{\mu,\alpha} = \frac{pn_{\mu,\alpha}-1}{p-1}$ ($\mu = 1, \dots, N^c$) denotes the α -component of the magnetic moment within C_{μ} . With increasing temperature T , the Potts model undergoes a phase transition at some critical temperature T_c from an ordered state, where $M \sim O(1)$ is strictly nonzero, to a disordered state with $M \simeq 0$. We use the similar s -CG approach to construct the CG-network with different N^c and compare the results obtained from CG-MC simulations with those of micro-MC simulations.

To begin, we show the results in Fig.1 for $\theta = 0$, where the network are essentially unweighted and the s -CG approach is identical to the d -CG . Fig.1(a) and 1(b) show the moment M and susceptibility $\chi = \beta N(\langle M^2 \rangle - \langle M \rangle^2)$

as functions of T , respectively. The susceptibility is related to the variance of the total magnetization according to the fluctuation-dissipation theorem. Apparently, our results (empty squares and solid circles) are in excellent agreements with the micro-level counterparts (solid lines). As comparisons, we have also shown the results obtained by a random-merging (RM) CG-model (dotted lines) and the heterogeneous mean field theories (HMFT)[34] (empty triangles). Here, the RM model means that one simply merge N/N^c randomly selected nodes to form a CG-node. Evidently this random scheme fails to reproduce the microscopic behaviors at all. The results of the HMFT are obtained by numerically solving the self-consistent equations of order parameter[34]. We find that the HMFT can predict the curve of $M \sim T$ quite well, however, it fails to predict the curve of $\chi \sim T$. Strikingly, even when the original network is reduced to one with only 16 CG-nodes, the CG model still faithfully reproduces the phase transition curves and fluctuation properties. Since N^c is largely reduced compared to N , a considerable speed-up of CPU time can be achieved which makes it feasible to study system size effects. Fig.1(c) plots T_c as a function of $\ln N$, obtained by our CG method with $N^c = 64$. T_c is determined as the location of the peak in the $\chi \sim T$ curve, see Fig.(1b). The dependence is linear with a slope $\simeq 1.68$, which agrees rather well with a theoretical prediction $T_c/\ln N = \frac{S}{4p} \simeq 1.67$ [34], where S is the average node strength in the network.

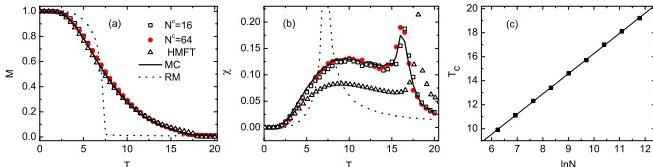


FIG. 1: (color online). (a)-(b) M and χ as functions of T (in unit of J/k_B) for the Potts model on unweighted SF networks ($\theta = 0$), obtained from brute-force MC simulation (solid line), HMFT(triangle), random-merging CG(dotted line), and the s -CG (square and circle). $N = 16384$, $p = 3$. (c) Dependence of T_c on the network size N obtained by the s -CG approach with fixed $N^c = 64$. All the networks have fixed mean degree $D = 20$. The error bars (not shown) are smaller than the symbol sizes.

For $\theta \neq 0$, the networks are weighted. Here we take $\theta = 2.4$ as an example to ensure the heterogeneity of the link weights. Figure(2a) and (2b) show M and χ as functions of T respectively. As in Fig.1(b), the peak in χ locates the critical point T_c . Clearly, the s -CG results (solid circle) are still in excellent agreements with the MC results (solid lines), however, the d -CG (solid squares) [18] and RM-CG (dotted lines) both fails. For such weighted networks, the dynamic equations of HMFT is not available either. Thus, for such weighted networks, our s -CG approach is the only promising CG approach so far. In Fig.2(c), we have also shown the de-

pendence of T_c on the network size. Apparently, there is also a linear dependence between T_c and $\ln N$ with the slope being about 1.288. As mentioned in the last paragraph, this slope depends on the average strength S . For a weighted network, one may estimate S by $\langle w_{ij} \rangle D$, where $\langle w_{ij} \rangle \simeq \int_0^2 x^\theta / 4dx = \frac{1}{4(\theta+1)} 2^{\theta+1}$. Substituting $D = 20$, $\theta = 2.4$ and $p = 3$ to these formula, we obtain $T_c/\ln N = \frac{S}{4p} \simeq 1.293$, which is consistent with the simulation value.

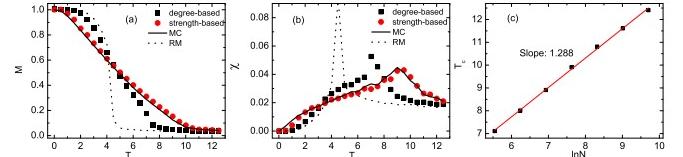


FIG. 2: (color online). (a)-(b) M and χ as functions of T for the ferromagnetic Potts model on a weighted scale-free network with mean degree $D = 20$. $N = 1024$, $p = 3$, $\theta = 2.4$ and $N^c = 16$. (c) Dependence of T_c on the network size N . The error bars are omitted for clarity since they are smaller than the symbol sizes.

In real-world networks, correlation is an ubiquitous feature. For instance, social networks show that nodes with large degrees tend to connect together, a property referred to as “assortative mixing”[35]. In contrast, many technological and biological networks show “disassortative mixing”, i.e., connections between high-degree and low-degree nodes are more probable [36, 37]. Previous studies showed that correlations may play important roles in network dynamics [35–39]. In the present work, we have used our s -CG method to study the phase transition of Potts model on correlated networks, which can not be studied by the HMFT which assumes no degree correlation. To characterize the assortative property of the weighted network, a strength correlation coefficient r , an extension of the degree correlation [35], can be defined as

$$r = (\langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle) / (\langle s_i^2 \rangle - \langle s_i \rangle^2). \quad (12)$$

Here s_i and s_j are the strengths of the two end-nodes of an edge. r is zero for networks with no strength-correlation, such as BA-SF networks, and positive or negative for assortative or disassortative networks, respectively.

Figure 3(a) shows T_c as a function of r , obtained from our s -CG approach and micro-MC simulations for $\theta = 0$. Again, the fits between CG-MC and MC are good. Figure 3(b) shows the effects of correlated network size on T_c . Interestingly, we find that the linear dependence between T_c and $\ln N$ is lost for correlated networks. For assortative(disassortative) networks T_c grows monotonically much faster(slower) than $\ln N$, respectively. In other words, the ordered state in an assortative(disassortative) network is harder(easier) to be destroyed with increasing temperature than in an un-correlated network. This is understandable since a ‘hub’-node in the network is

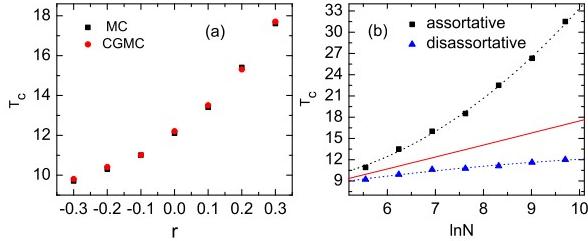


FIG. 3: (color online). Phase transition behaviors of the Potts model on unweighted correlated networks. (a) T_c plotted as a function of the network correlation coefficient r , obtained via CG-MC and MC simulations. $N = 1024$ and $N^c = 64$. (b) Dependence of T_c on the network size N . All the networks have fixed mean degree $D = 20$.

more difficult to change its state than a ‘leaf’-node due to larger energy barrier. In an assortative network, hub-nodes are connected together, such that they tend to freeze into a local ordered state which is stable to thermal fluctuations. For a disassortative network, a hub-node is usually connected to many leaf-nodes. Since leaf-nodes can change state easily, the ‘alone’ hub-node is more likely to change state with the help of their ‘boiling’ neighbors. Therefore, assortative correlations tend to increase T_c as observed here.

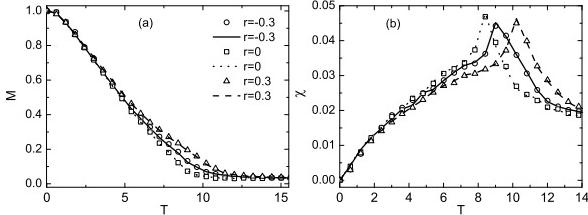


FIG. 4: Magnetization M and susceptibility χ as functions of temperature T for the Potts model on weighted correlated networks. Symbols and lines correspond to the CG-MC and micro-MC simulation results, respectively. Other parameters are same as in Fig.2.

In Fig.(4), the magnetization M and susceptibility χ of the ferromagnetic Potts model on weighted networks are plotted as functions of temperature T at different correlation coefficient r , obtained from our s -CG approach and micro-MC simulations. Again, the agreements between CG-MC and MC are excellent, further demonstrating the validity of our method.

IV. CONCLUSIONS

In summary, we have developed a strength-based s -CG approach for coarse-graining study of the phase transition of the Potts model on weighted networks. We have utilized a mean-field scheme to generate the connectivity of the CG-network and derived the CG-Hamiltonian. To address the problem how to guarantee the validity of the CG-model, we have proposed the so-called CSC, which requires that the probability to find a given CG-configuration in the equilibrium state, calculated from the CG-model, should be the same as that calculated from the original microscopic model. We show, by performing error analysis, that our s -CG approach, by merging nodes with close strengths together, holds the CSC approximately with ANA. Detailed numerical simulations demonstrate clearly that our s -CG approach can reproduce the microscopic MC simulation results very well, not only for the onset of phase transition, but also for the fluctuations and system size effects.

Acknowledgments

This work was supported by the National Science Foundation of China under Grants No.20933006 and No.20873130.

-
- [1] R. Albert and A.-L. Barabási, Rev. Mod. Phys. **74**, 47 (2002).
 - [2] S. N. Dorogovtsev and J. F. F. Mendes, Adv. Phys. **51**, 1079 (2002).
 - [3] M. Newman, SIAM Review **45**, 167 (2003).
 - [4] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, and D.-U. Hwang, Phys. Rep. **424**, 175 (2006).
 - [5] A. Arenas, A. Díaz-Guilera, J. Kurths, Y. Moreno, and C. Zhou, Phys. Rep. **469**, 93 (2008).
 - [6] D. P. Landau and K. Binder, *A Guide to Monte Carlo Simulations in Statistical Physics* (Cambridge University Press, Cambridge, 2000).
 - [7] M. I. Rabinovich, P. Varona, A. I. Selverston, and H. D. I. Abarbanel, Rev. Mod. Phys. **78**, 1213 (2006).
 - [8] B. J. Kim, Phys. Rev. Lett. **93**, 168701 (2004).
 - [9] C. Song, S. Havlin, and H. A. Makse, Nature **433**, 392 (2005).
 - [10] K.-I. Goh, G. Salvi, B. Kahng, and D. Kim, Phys. Rev. Lett. **96**, 018701 (2006).
 - [11] F. Radicchi, J. J. Ramasco, A. Barrat, and S. Fortunato, Phys. Rev. Lett. **101**, 148701 (2008).
 - [12] D. Gfeller and P. D. L. Rios, Phys. Rev. Lett. **100**, 174104 (2008).
 - [13] I. G. Kevrekidis, C. W. Gear, J. M. Hyman, P. G. Kevrekidis, O. Runborg, and C. Theodoropoulos, Comm. Math. Sci. **1**, 715 (2003).
 - [14] S. J. Moon, R. Ghanem, and I. G. Kevrekidis, Phys. Rev. Lett. **96**, 144101 (2006).
 - [15] R. Erbana, I. G. Kevrekidis, D. Adalsteinsson, and T. C. Elston, J. Chem. Phys. **124**, 084106 (2006).

- [16] T. Gross and I. G. Kevrekidis, Eur. Phys. Lett. **82**, 38004 (2008).
- [17] S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, Rev. Mod. Phys. **80**, 1275 (2008).
- [18] H. S. Chen, Z. H. Hou, H. W. Xin, and Y. J. Yan, Phys. Rev. E **82**, 011107 (2010).
- [19] M. E. J. Newman, Phys. Rev. E **64**, 016131 (2001).
- [20] M. E. J. Newman, Phys. Rev. E **64**, 016132 (2001).
- [21] A. L. Barabasi, H. Jeong, Z. Nda, E. Ravasz, A. Schubert, and T. Vicsek, Phys. Rev. E **311**, 590 (2002).
- [22] A. Barrat, M. Barthlemy, R. Pastor-Satorras, and A. Vespignani, Proc. Natl. Acad. Sci. USA **101**, 3747 (2004).
- [23] E. Almaas, P. L. Krapivsky, and S. Redner, Phys. Rev. E **71**, 036124 (2005).
- [24] E. Almaas, B. Kovcs, T. Viscek, Z. N. Oltval, and A. L. Barabasi, Nature (London) **427**, 839 (2004).
- [25] S. L. Pimm, *Food Webs, 2nd ed.* (The University of Chicago Press, Chicago, 2002).
- [26] F. Y. Wu, Rev. Mod. Phys. **54**, 235 (1982).
- [27] M. A. Garey and D. S. Johnson, *Computers and intractability* (Freeman, New York, 1979).
- [28] S. H. Yook, H. Jeong, and A.-L. Barabasi, Phys. Rev. Lett. **86**, 5835 (2001).
- [29] M. Boguñá and R. Pastor-Satorras, Phys. Rev. E **68**, 036112 (2003).
- [30] G. Caldarelli, A. Capocci, P. D. L. Rios, and M. A. Moreno, Phys. Rev. Lett. **89**, 258702 (2002).
- [31] R. Pastor-Satorras and A. Vespignani, Phys. Rev. E **65**, 035108R (2002).
- [32] A.-L. Barabási and R. Albert, Science **286**, 509 (1999).
- [33] K. Park, Y. C. Lai, and N. Ye, Phys. Rev. E **70**, 026109 (2004).
- [34] S. Dorogovtsev, A. Goltsev, and J. Mendes, Eur. Phys. J. B **38**, 177 (2004).
- [35] M. E. J. Newman, Phys. Rev. Lett. **89**, 208701 (2002).
- [36] R. Pastor-Satorras, A. Vázquez, and A. Vespignani, Phys. Rev. Lett. **87**, 258701 (2001).
- [37] S. Maslov and K. Sneppen, Science **296**, 910 (2002).
- [38] A. V. Goltsev, S. N. Dorogovtsev, and J. F. F. Mendes, Phys. Rev. E **78**, 051105 (2008).
- [39] M. Boguna and R. P. Satorras, Phys. Rev. E **66**, 047104 (2002).